Lecture 12: Integrals and the Fundamental Theorem of Calculus

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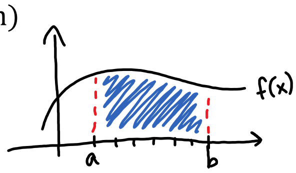
REVIEW of material covered so much

1) derivatives
2) antiderivatives
   a. F(x) is an antiderivative of f(x) if $\frac{d}{dx}F(x) = f(x)$
3) area under a curve:
   a. definite integrals (Riemann sum)

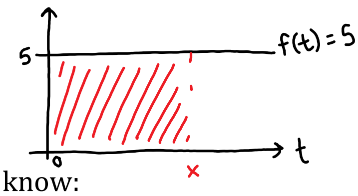
$$R = \lim_{n\to\infty}\left(\sum_{i-1}^{n} f(x_i) * \underset{\text{width}}{\Delta x}\right)$$



5.3

NOTICE: derivatives and integrals are inverses of each other

Ex:

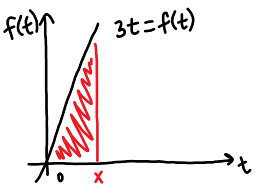


$$A = \underset{\text{height}}{5}\underset{\text{width}}{x}$$

We know:

$$F(x) = \int_0^x f(t)\,\mathrm{dt} = \int_0^x 5\,\mathrm{dt} = 5x$$

Ex 2:



$$A = \frac{bh}{2} = \frac{x * 3x}{2}$$

$$\int_0^x f(t)\,\mathrm{dt} = \frac{3x^2}{2}$$

$$\int_0^x 3t\,dt = \frac{3x^2}{2}$$

what happens if we differentiate this?

$$\frac{d}{dx}\left(\frac{3x^2}{2}\right) = \frac{3}{2}\frac{d}{dx}(x^2) = \frac{3}{\cancel{2}}\cancel{2}x = 3x$$

Fundamental Theorem of Calculus - part 1

If f is continuous on [a,b], then:

$$g(x) = \int_a^x f(t)\,\mathrm{dt} \qquad a \le x \le b$$

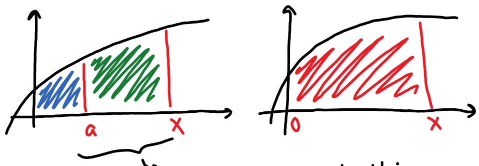
is continuous on [a,b] and differentiable on (a,b) and g'(x)=f(x)

other notation:

$$\frac{d}{dx}\int_a^x f(t)\,\mathrm{dt} = f(x)$$

Why $\int_a^x f(t)dt$ and not $\int_0^x f(t)dt$?

Why $\int_a f(t)dt$ and not $\int_0 f(t)dt$?



we can compute this as

$$\int_0^x f(t)dt - \int_0^a f(t)dt$$

If F(x)=$\int_0^x f(t)dt$ is the same as F(x)-F(a)

F is an antiderivative of f, but so is F(x)-F(a) because:

$$\left(F(x) - \underset{\text{constant}}{F(a)}\right)' = F'(x) = f(x)$$

The Fundamental Theorem of Calculus - part 2

If f is continuous on [a,b], then:

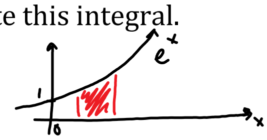
$$\int_a^b f(x)dx = F(b) - F(a)$$ where F(x)is an antiderivative of f(x)

such that F'(x)=f(x)

connection between area under f(x) and its antiderivative!

<u>Ex</u>: Evaluate this integral.

$$\int_1^3 e^x\, dx$$



Antiderivative of e^x is e^x

So:

$$\int_1^3 e^x\, dx = F(3) - F(1)$$

where F(x)= e^x, an antiderivative of f(x).

$= e^3 - e^1$

NOTE:

What if we choose $F(x) = e^x + 7$ as antiderivative?

Then,

$$\int_1^3 e^x\, dx = F(3) - F(1) = (e^3 + 7) - (e^1 + 7) = e^3 - e$$

ie. It does not matter which antiderivative we choose!

<u>Ex</u>:

$$\int_3^6 \frac{dx}{x} = F(6) - F(3) = \ln(|6|) - \ln(|3|) = \ln(2)$$

What is an antiderivative of $f(x) = \frac{1}{x}$?

$F(x) = \ln(|x|)$

Integrals with definite boundaries are *definite integrals*.

<u>5.4 Indefinite integrals</u> (no bounds)

$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x)$$

(F(x) is an antiderivative of f(x))

then, for example:

$$\int x^2\,\mathrm{d}x = \frac{x^3}{3} + C$$

For indefinite integrals, you **need** +C

- they are essentially antiderivatives

Ex:

$$\left(\frac{x^3}{3} + C\right)' = \left(\frac{x^3}{3}\right)' + (C)' = \frac{3x^2}{3} + 0 = \frac{3x^2}{3}$$

A few important integrals (see full table in textbook):

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad \int \mathrm{e}^x\,dx = e^x + C$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \arctan(x) = \tan^{-1} x$$

$$\int sec^2(x)\,dx = \arccos(x) + C$$

Ex:

$$\int (10x^4 - 2\sec^2(x))\,dx$$

$$\int 10x^4\,dx - \int 2\sec^2(x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10 * \frac{x^5}{5} - 2\tan(x) + C$$

only one + C for both integrals

Ex:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$$

$$= \int_1^9 \left(2 + \sqrt{t} - \frac{1}{t^2}\right) dt$$

$$= 2t + \frac{2}{3}t^{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9 = 2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\Bigg]_1^9$$

evaluate between boundaries

F(b)-F(a)
F(9)-F(1)

$$= 2(9) + \frac{2}{3}9^{\frac{3}{2}} + \frac{1}{9} - \left(2(1) + \frac{2}{3}(1)^{\frac{3}{2}} + 1^{-1}\right)$$

$$= \frac{292}{9} \approx 32.44$$

<u>Application</u> in physics

- determining distance travelled by a falling object in 5s

Know:

$a(t) = 9.81\,\frac{m}{\ldots}$ distance

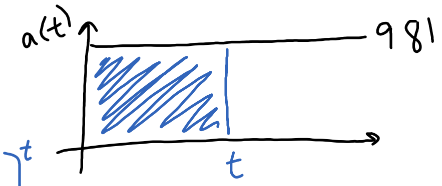
- determining distance travelled by a falling object in 5s

Know:

$$a(t) = 9.81\frac{m}{s^2}$$

distance

$$v(t) = \int a(t)\,dt, \qquad s(t) = \int v(t)\,dt$$



$$v(t) = \int_0^t a(x)\,dx$$

$$= \int_0^t 9.81\,dx = 9.81x\Big|_0^t$$

$$= 9.81t - 9.81(0) = 9.81t$$

$$s(t) = \int_0^t v(x)\,dx = \int_0^t 9.81x\,dx = 9.81\frac{x^2}{2}\Big|_0^t$$

$$= 9.81\frac{t^2}{2} - 9.81 * \frac{0^2}{2} = 9.81\frac{t^2}{2}$$ function for distance

at time t=5: $s(5) = 9.81\left(\frac{5^2}{2}\right) = 122.625m$